

Abstract

The purpose of this note is to rephrase Speyer's elegant topological proof for Kasteleyn's Theorem in a simple graph theoretical manner.

Speyer’s elegant topological proof for Kasteleyn’s Theorem

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1 Introduction

Speyer [2] recently published a very short and elegant proof for Kasteleyn’s Theorem. This proof involves some “higher topological arguments”; and the purpose of this note is to replace these by elementary graph-theoretical arguments. (Unfortunately, this simplification makes the presentation significantly longer.)

This note is organized as follows: Section 2 contains background information, Section 3 rephrases Speyer’s main argument in “simple graph-theoretical language”, Section 4 works out the details of the proof, and Section 5 presents an illustrating example. (A time-efficient way to read through this note would start at the end of Section 2 for recalling (the slight generalization of) Kasteleyn’s Theorem, followed by a look at the example in Section 5 for getting the main idea of the proof.)

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2 Notation and background information

For reader's convenience and for fixing our notation, we recall some background information on multisets, graphs, weights, matchings, generating functions and Pfaffians.

Finite sets and multisets

We shall use the shorthand notation $[n] := \{1, 2, \dots, n\} \subset \mathbb{N}$ for the first n natural numbers (not to be confused with the standard notation $[t_0, t_1] = \{x \in \mathbb{R} : t_0 \leq x \leq t_1\}$).

If S is some set, then we shall consider finite *multi(sub)sets* X of S , which may contain several copies of elements of S , and we denote this by $X \sqsubseteq S$. For instance,

$$\{1, 3, 3, 4\} \sqsubseteq \mathbb{N}$$

is a multiset of natural numbers containing two copies of the number 3, which we shall also denote as

$$\{1, 3^{[2]}, 4\}.$$

If a multiset X contains precisely k copies of some element x , then we say “ k is the *multiplicity* of x in X ”. (So “the multiplicity of x in X is 0” is another way to express that $x \notin X$.) The *disjoint union* $X \dot{\cup} Y$ of two multisets $X, Y \sqsubseteq S$ is defined as the multiset containing all elements $x \in A \cup B$ (usual union of sets) with multiplicity equal to the sum of the multiplicities in A and in B . For instance,

$$\{1, 3^{[2]}, 4\} \dot{\cup} \{2, 3, 4^{[2]}, 5\} = \{1, 2, 3^{[3]}, 4^{[3]}, 5\} \sqsubseteq \mathbb{N}.$$

If S is some finite set, then we denote the family of all subsets of S with k elements by

$$\binom{S}{k} := \{X \in S : |X| = k\}.$$

Finite graphs

Throughout this note, let $G = (V(G), E(G))$ be a *finite graph* on n *labeled* vertices and m *labeled* edges, i.e.,

- $V(G) = \{v_1, v_2, \dots, v_n\}$ is a finite set of elements called the *vertices* of G ,
- $E(G) = \{e_1, e_2, \dots, e_m\} \subseteq \left(\binom{V(G)}{2} \cup \binom{V(G)}{1} \right)$ is a *multiset* of 1–element or 2–element subsets of $V(G)$, which are called the *edges* of G .

If some vertex v belongs to some edge e (i.e., $v \in e$), then v and e are said to be *incident* to one another. If $e_k = \{v_i, v_j\} \in E(G)$, then we say that v_i and v_j are *adjacent*, or e_k *joins* v_i and v_j : In this case, we shall also denote this edge by $e_{i,j}$ (or, equivalently, by $e_{j,i}$).

An edge $e = \{v_i\} \in E(G)$ joins vertex v_i with itself: Such edge is called a *loop*; and a graph with no loops is called *loopless*. Since $E(G)$ is a *multiset*, there might be more than one edge connecting vertices v_i and v_j : Such edges are called *multiple edges*. A loopless graph without multiple edges is called *simple*.

The *degree* $\deg(v)$ of a vertex v is the number of non–loop edges it is incident with, plus *twice* the number of loops it is incident with.

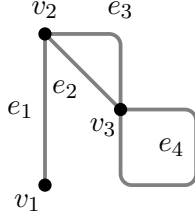
A sequence of vertices $(v_{i_0}, v_{i_1}, \dots, v_{i_k})$ is called a *trail* of length k which connects v_{i_0} with v_{i_k} , if $\{v_{i_{j-1}}, v_{i_j}\} \in E(G)$ for $j = 1, 2, \dots, k$. If all the vertices $v_{i_0}, v_{i_1}, \dots, v_{i_k}$ are *distinct* with the possible exception $v_{i_0} = v_{i_k}$, then the trail is called a *path*; if $v_{i_0} = v_{i_k}$, then the path is called a *cycle* of length k . (A loop is a cycle of length 1.)

If the vertex set $V(G)$ of a graph G can be written as the disjoint union $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that there is *no edge* joining a vertex from V_1 with a vertex from V_2 , then G is called a *bipartite graph*. Clearly, a bipartite graph must be loopless. More generally, it is a well–known fact that a graph is bipartite if and only if it has no cycle of odd length.

As the term “graph” suggests, there is an obvious *graphical representation* of a graph G , where vertices are represented by points and edges are represented by curves connecting these points, see Figure 1.

Figure 1: Graphical representation of graphs.

The picture shows the graphical representation of a graph G with three vertices v_1, v_2, v_3 and four edges e_1, e_2, e_3, e_4 , where e_4 is a loop (connecting v_3 with itself), and e_2 and e_3 are multiple edges (both connecting v_2 with v_3).



Weighted edges

Throughout this note, let $\omega : E(G) \rightarrow R \setminus \{0\}$ be a (nowhere-zero) *weight function*, called *edge weight*, on the multiset of edges of G , where R is some (nontrivial) commutative ring (in most cases, R is \mathbb{Z} or some polynomial ring; the constant edge weight $\omega \equiv 1$ is used for enumeration problems).

If $e = \{v_i, v_j\}$, we shall denote the weight of e also by $\omega(e) = \omega_{i,j} = \omega_{j,i}$.

The weight of a subset $S \subseteq E(G)$ is the product of the weights of the edges in S :

$$\omega(S) := \prod_{e \in S} \omega(e).$$

Properly intersecting curves

In order to rule out monstrosities like Peano-curves (which clearly never occur when we *actually draw* some graph G on a sheet of paper), we need to be more restrictive on the points and curves used for graphical representations (see Figure 1).

Let $T = \{\tau_1, \dots, \tau_m\}$ be a finite set of *differentiable* plane curves

$$\tau_i : [0, 1] \rightarrow \mathbb{R}^2 \text{ for } i \in [m]$$

with *nowhere vanishing* derivatives (i.e., $\dot{\tau}_i(t) \neq (0, 0)$ for all $(i, t) \in [m] \times [0, 1]$).

We say that $\tau_i(0)$ and $\tau_i(1)$ are the *endpoints* of τ_i , and we denote the *set of endpoints* of all curves in T by

$$\text{end}(T) := \{\tau_i(t) : i \in [m], t \in \{0, 1\}\}.$$

We say that τ_i *passes through* $p \in \mathbb{R}^2$ if there exists some t , $0 < t < 1$, with $\tau_i(t) = p$. A curve might pass through some point p more than once, and two or more curves might pass through the same point p : A *passage of some curve τ_i through p* is simply a real number $t \in (0, 1)$ such that $\tau_i(t) = p$. For any given point p , denote

- the set of *all* curves in T which pass through p by $\text{crv}(p, T)$,
- the set of *all* passages of curve τ_i through p by $\text{psg}(p, \tau_i)$,

(these sets, of course, can be empty). A point p with $|\text{crv}(p, T)| > 2$ (i.e., at least two *different* curves pass through p) or with $|\text{psg}(p, \tau_i)| > 2$ for some i (i.e., curve τ_i passes through p at least *twice*) is called an *intersection point* of T .

We call the set τ *properly intersecting* if

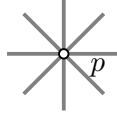
1. $\tau_i(t) \notin \text{end}(T)$ for all $i \in [m]$, $t \in (0, 1)$, i.e., no curve passes through an endpoint,
2. there are only finitely many intersection points,
3. for every intersection point p and every τ in T , the set $\text{psg}(p, \tau)$ of passages of curve τ through p is finite,
4. and for each two *different* passages t, s of curves τ_i, τ_j through an intersection point p (i.e., $\tau_i(t) = \tau_j(s) = p$, where $i \neq j$ or $s \neq t$), the derivatives $\dot{\tau}_i(t)$ and $\dot{\tau}_j(s)$ are *linearly independent*.

Note that the third condition for properly intersecting curves implies that k passages through some point p *locally* resemble the intersection of k *different* straight lines in a single point of the plane, see Figure 2.

Since the actual parametrization of the curves τ_i is not important in the following, we shall (sloppily) identify the curve τ_i with its image $\tau_i([0, 1])$ from now on.

Figure 2: Situation near an intersection point p of properly intersecting curves.

The picture shows 4 passages of properly intersecting curves through point p .



Proper drawings

It is easy to see that a finite simple graph G can always be *drawn* as (or “represented” by) a properly intersecting set T of curves in the following sense:

- Every *vertex* v_i of G is drawn as (or “represented by”) a unique *endpoint* of some $\tau \in T$, i.e., there is a bijection

$$\mu : V(G) \rightarrow \text{end}(T).$$

- Every edge $e_k = \{v_i, v_j\}$, $i < j$, is drawn as (or “represented by”) a curve $\tau_k : [0, 1] \rightarrow \mathbb{R}^2$ such that $\tau_k(0) = \mu(v_i)$ and $\tau_k(1) = \mu(v_j)$.

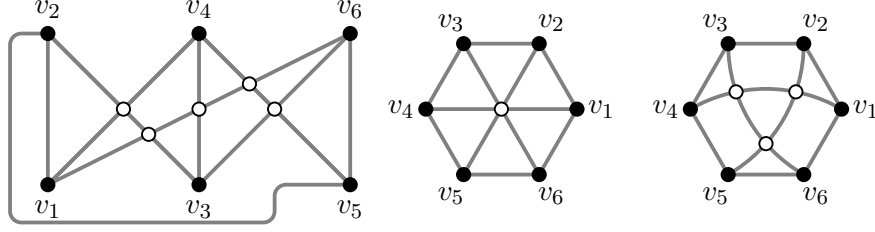
We shall call such graphical representation of a graph G a *proper drawing* of G and denote it by $(G; T)$. See Figure 3 for an illustration: Throughout this note, when showing a proper drawing $(G; T)$, we shall indicate *vertices* by small black circles and *intersection points* by small white circles.

Stembridge’s proper drawing

For every finite simple graph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ there is a specific proper drawing $(G; \overline{T})$, which we shall call *Stembridge’s proper drawing* (see [3]): Every edge $e = e_{i,j} = \{v_i, v_j\}$, $i < j$, is represented by the half-circle in the upper half-plane with center $(\frac{i+j}{2}, 0)$ and radius $\frac{|i-j|}{2}$, i.e.,

$$\overline{\tau}_e(t) := \left(\frac{i+j}{2} + \frac{|i-j|}{2} \cdot \cos(\pi \cdot (1-t)), \frac{|i-j|}{2} \cdot \sin(\pi \cdot (1-t)) \right);$$

Figure 3: Three proper drawings of the complete bipartite graph $K_{3,3}$. The 6 vertices of $G = K_{3,3}$ are indicated by black circles, intersection points are indicated by white circles. Recall that $K_{3,3}$ is not planar, so there is no proper drawing without an intersection point.



which implies that every vertex v_i is represented by the point $\mu(v_i) = (i, 0)$. See the left picture in Figure 5 for an example of this specific proper drawing.

Faces of a proper drawing

Given a proper drawing $(G; T)$, we consider the plane with all images of the curves τ removed, i.e.,

$$\mathbb{R}^2 \setminus \left(\bigcup_{k=1}^m \{ \tau_k([0, 1]) \} \right).$$

We call the connected components of this set the *faces* of the proper drawing and denote the set of these faces by $F(G; T)$.

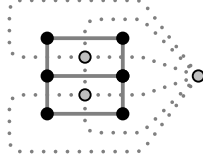
Planar drawings

If there exists a proper drawing $(G; T)$ of G such that there are *no intersection points at all*, then G is called a *planar graph*, and we call $(G; T)$ a *planar drawing*.

If $(G; T)$ is a planar drawing, then we may consider its *dual graph* \hat{G} with

$$V(\hat{G}) = F(G; T), E(\hat{G}) = E(G);$$

Figure 4: A proper drawing of a simple planar graph and its dual.
The picture shows a planar drawing of a simple planar graph on 6 vertices and 3 faces, represented in the picture by small gray circles. The dual graph is not simple; its edges are represented in the picture by dotted lines.



where an edge e of \hat{G} joins two faces f_i and f_j if τ_e belongs to the intersection of the boundaries of f_i and f_j in $(G; T)$. If $e_k = \{f_i, f_j\} \in E(\hat{G})$, then we say that f_i and f_j are *adjacent*.

Choose a point $\hat{\mu}(f) \in f$ for every $f \in F(G; T)$: It is easy to see that we can construct a set of curves $\{\hat{\tau}_e : e \in E(G)\}$ such that $(\hat{G}; \hat{\tau})$ is a planar drawing, with the additional properties that

$$\{\tau_e : e \in \mathbb{E}(G)\} \cup \{\hat{\tau}_e : e \in \mathbb{E}(G)\}$$

is properly intersecting , and

$$\chi(\{\tau_{e_i}, \hat{\tau}_{e_j}\}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

see Figure 4.

Observe that *every* proper drawing $(G; T)$ *appears* as a planar drawing $(G'; \tau')$ if we *reinterpret* all intersection points of the drawing as vertices of some graph G' and define the edges τ' of G' to be appropriate *segments* of the edges τ of $(G; T)$; note that all “reinterpreted intersection points” have *even degree* in G' .

Crossings of edges

Let $(G; T)$ be a proper drawing of G , let $\tau_i, \tau_j \in T$, be the two curves representing the (different) edges e_i, e_j , $i \neq j$. Each passage of τ_i and τ_j through a common intersection point p is called a *crossing*; i.e., the number of crossings of e_i, e_j in p is

$$\chi(e_i, e_j, p; T) = |\text{psg}(p, \tau_i)| \cdot |\text{psg}(p, \tau_j)|,$$

and the number of *all* crossings of e_i, e_j is

$$\chi(e_i, e_j; T) = \sum_p \chi(e_i, e_j, p; T)$$

where the sum ranges over all intersection points p of T .

Signed subsets of edges

Consider some fixed proper drawing $(G; T)$. For every subset $S = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_k\} \subseteq E(G)$, the number of *all* crossings of edges in S is

$$\chi(S; T) = \sum_{1 \leq i < j \leq k} \chi(\bar{e}_i, \bar{e}_j, T).$$

We define the *sign* of S (with respect to $(G; T)$) as

$$\text{sgn}(S; T) := (-1)^{\chi(S; T)}.$$

Perfect matchings

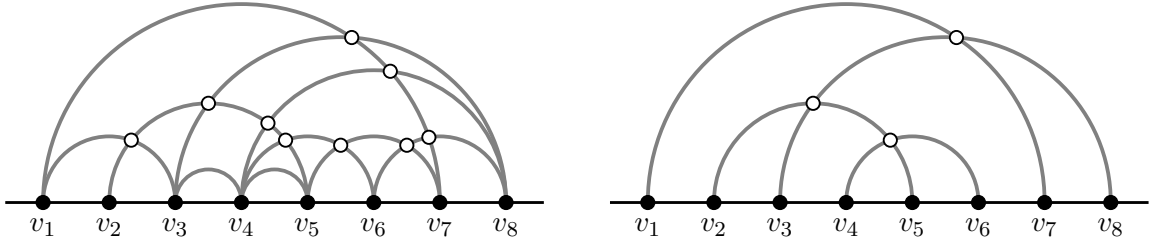
Let G be a finite *simple* graph. A *perfect matching* $M \subseteq E(G)$ of G is a subset of edges such that every vertex of G is contained in precisely one edge of M . The *generating function* $m(G, \omega)$ is defined as

$$m(G, \omega) := \sum_M \omega(M),$$

where M ranges over all perfect matchings of G .

Figure 5: Stembridge's proper drawing and the Pfaffian.

The left picture shows Stembridge's proper drawing $(G; \overline{T})$ of a graph G . Observe that the four edges $e_{1,7}, e_{2,5}, e_{3,8}$ and $e_{4,6}$ constitute a perfect matching M of G , which is shown in the right picture: We have $\chi(M; \overline{T}) = 3$, so M corresponds to the summand $-\omega_{1,7} \cdot \omega_{2,5} \cdot \omega_{3,8} \cdot \omega_{4,6}$ in the Pfaffian $\text{Pf}(G, \omega)$.



For some fixed proper drawing $(G; T)$ of G , we consider also the *signed generating function* $s(G, \omega; T)$ which is defined as

$$s(G, \omega; T) := \sum_M \text{sgn}(M; T) \cdot \omega(M),$$

where M ranges over all perfect matchings of G .

Clearly, $m(G, \omega) = s(G, \omega; T) \equiv 0$ if $n = |V(G)|$ is an *odd* number.

Pfaffians

The *Pfaffian* of a graph G is defined as the signed generating function for Stembridge's proper drawing $(G; \overline{T})$, i.e.,

$$\text{Pf}(G, \omega) := s(G, \omega; \overline{T}).$$

Sign-modifications of edge weights

Let G be some simple graph with edge weight ω : Another edge weight ω' is called a *sign-modification* of ω if for all $e \in E(G)$ there holds

$$\omega(e) = \pm \omega'(e).$$

The purpose of this note is to rephrase Speyer's proof [2] for Kasteleyn's Theorem [1].

Theorem 1 (Kasteleyn's Theorem). *Let G be a planar finite simple graph on n labeled vertices with edge weight ω . Let $(G; T)$ be an arbitrary proper drawing of G . Then there exists a sign-modification ω' of ω such that*

$$m(G, \omega) = \text{Pf}(G, \omega').$$

In fact, we shall proof a slight generalization:

Theorem 2. *Let G be a finite simple graph on n labeled vertices with edge weight ω . Let $(G; T)$ be an (arbitrary, but fixed) proper drawing of G . Then for every proper drawing $(G; T')$ there is a sign-modification ω' of ω such that*

$$s(G, \omega; T) \equiv s(G, \omega'; T').$$

3 Main argument in Speyer's proof

Think of the edges of graph G as a set of “infinitely thin, ductile and flexible” strings tied together in the vertices of G , and view a proper drawing (G, T) as an arrangement of such “web of strings” in the plane (with certain conditions on crossings). Strings and vertices can be “dragged around” in order to obtain another proper drawing (G, T') : In the process of such movements, of course, new crossings might be introduced and existing crossings might be removed.

In the following, we will view the proper drawing as a *planar* drawing. We will consider two *basic moves* involving

- some vertex \bar{v} and some face $f_{\bar{v}}$ which contains \bar{v} in its boundary,

- some edge \bar{e} and some face $f_{\bar{e}}$ which contains a *segment* of \bar{e} in its boundary,
- a *path* p in the *dual graph* connecting $f_{\bar{e}}$ and $f_{\bar{v}}$.

The picture in the middle of Figure 6 gives an illustration.

3.1 Drag edge \bar{e} over vertex \bar{v}

Observe that we can “drag” the (segment of the) edge \bar{e} along the path p , introducing

- *precisely two* new crossings with all (segments of) edges belonging to p ,
- and *precisely one* new crossing with all (segments of) edges incident to \bar{v} ,

and not introducing or removing any other crossings. The meaning of this should become clear when looking at the left picture in Figure 6.

Now the point is that we can *offset* the effect of these new crossings on the signed weights

$$\text{sgn}(M; T) \cdot \omega(M)$$

of *perfect matchings* M of G by letting

$$\omega'(e) = \begin{cases} \omega(e) & \text{if } e \neq \bar{e}, \\ -\omega(e) & \text{if } e = \bar{e}, \end{cases}$$

since every perfect matching M must contain *precisely one* of the edges *incident* with \bar{v} :

- If $\bar{e} \notin M$, then we have $\text{sgn}(M; T) = \text{sgn}(M; T')$ and $\omega(M) = \omega'(M)$,
- if $\bar{e} \in M$, then we have $\text{sgn}(M; T) = -\text{sgn}(M; T')$ and $\omega(M) = -\omega'(M)$,

whence altogether we have

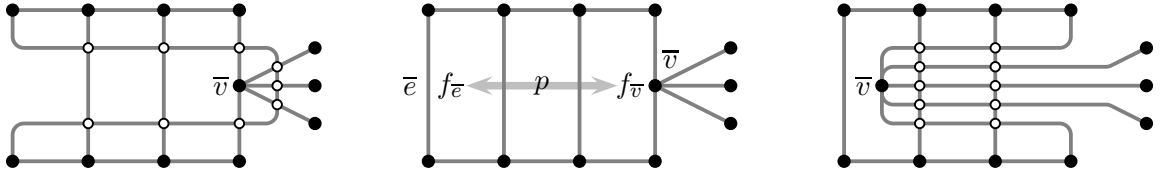
$$s(G, \omega; T) = s(G, \omega'; T').$$

Figure 6: The main idea of the proof.

The picture in the middle shows a proper drawing of some (part of a) graph G , viewed as a planar graph (so points shown here as vertices might actually be crossings, and edges might actually be segments of edges): Note that edge \bar{e} belongs to the boundary of face $f_{\bar{e}}$, and vertex \bar{v} belongs to the boundary of face $f_{\bar{v}}$. A double-tipped arrow indicates a path p connecting $f_{\bar{v}}$ and $f_{\bar{e}}$ in the dual graph.

The left picture shows the result of “dragging” the edge \bar{e} over the vertex \bar{v} along p .

The right picture shows the result of “dragging” the vertex \bar{v} into the face $f_{\bar{e}}$ along p .



3.2 Drag vertex \bar{v} into face \bar{f}

Observe that we can “drag” the vertex \bar{v} into face \bar{f} along path p , introducing

- *precisely one* new crossing for
 - all (segments of) edges incident with \bar{v}
 - with all (segments of) edges belonging to p ,

and not introducing or removing any other crossings. The meaning of this should become clear when looking at the right picture in Figure 6.

Now by the same reasoning as above we see that we can *offset* the effect of these new crossings on the signs of *perfect matchings* of G by letting

$$\omega'(e) = \begin{cases} \omega(e) & \text{if } e \text{ does not belong to } p, \\ -\omega(e) & \text{if } e \text{ belongs to } p. \end{cases}$$

Our strategy for proving Theorem 2 is to “transform” the proper drawing (G, T) to the proper drawing (G, T') by dragging vertices and edges and

offsetting all sign changes for perfect matches by the corresponding sign-modifications of the weight function; as described above.

4 Details of the proof

We shall decompose the proof of Theorem 2 into two steps, expressed in Propositions 1 and 2:

Proposition 1. *Let G be a finite simple graph on n labeled vertices $\{v_1, \dots, v_n\}$ with edge weight ω . Let $(G; T)$ be a proper drawing of G , and let $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$ be a set of n points. Then there exists a proper drawing $(G; T'')$ with $\mu(v_i) = p_i$ (i.e., vertex v_i is represented by point p_i) and a sign-modification ω'' such that*

$$s(G, \omega; T) \equiv s(G, \omega''; T'').$$

Proof. We may assume that P is a subset of the *unbounded* face f_1 of the proper drawing $(G; T)$ (if this is not the case, we may *move* P there by a simple translation, construct the proper drawing as claimed, and move this drawing back by the inverse translation).

It is easy to see that we can drag the point $\mu(v_1)$ (corresponding to v_1) to p_1 in this face f_1 such that each of the remaining points p_2, \dots, p_n belongs to some face of the proper drawing obtained by this operation. Moreover, there is a sign-modification for the weight function that offsets all sign-effects introduced by this operation. (See the description in section 3.2.)

In particular, p_2 now belongs to some face f_2 , and we can drag point $\mu(v_2)$ to p_2 in f_2 and apply the appropriate sign-modification as before. Repeating this step, we finally arrive at a proper drawing $(G; T'')$ where vertex v_i is represented by p_i and $s(G, \omega; T) \equiv s(G, \omega''; T'')$, as claimed. \square

Proposition 2. *Let G be a finite simple graph on n labeled vertices $\{v_1, \dots, v_n\}$ with edge weight ω'' . Let $(G; T'')$ and $(G; T')$ be two proper drawings such that every vertex v_i is represented by the same point in both drawings, i.e.,*

$$\mu''(v_i) = \mu'(v_i) \text{ for all } i \in [n].$$

Then there exists a sign-modification ω' of ω'' such that

$$s(G, \omega''; T'') \equiv s(G, \omega'; T').$$

The idea of proof is: “Drag edges over vertices (and modify the weight function accordingly) until the drawing $(G; T')$ is reached”. We shall try to make this precise as follows.

Double edge

Consider the (non-simple) finite graph D with $V(D) = \{v_1, v_2\}$ and $E(D) = \left\{ \{v_1, v_2\}^{[2]} \right\}$, i.e., $E(D)$ is the multiset consisting of *two* copies $e_1 = e_2 = \{v_1, v_2\}$: We call such graph D a *double edge*.

Proper colouring of the faces of a double edge

Let $(D; T)$ be a proper drawing of a double edge. View this drawing as a *planar drawing* (i.e., reinterpret intersection points as vertices) of some planar graph G . Observe that *every* vertex in G has *even* degree: It is easy to see that this implies that every cycle of the dual graph \hat{G} (with respect to this planar drawing) has *even* length, whence \hat{G} is *bipartite*. So we may colour the faces of D with two colours such that no two adjacent faces have the same colour: Call this a *proper colouring* of the faces of $(D; T)$.

Decomposition of crossings of three curves

Consider a properly intersecting set of three curves $T = \{\tau_1, \tau_2, \sigma\}$. By definition, for every intersection point p of T we have

$$\begin{aligned} \chi(\tau_1, \sigma, p; T) + \chi(\tau_2, \sigma, p; T) &= |\text{psg}(p, \tau_1) \times \text{psg}(p, \sigma)| + |\text{psg}(p, \tau_2) \times \text{psg}(p, \sigma)| \\ &= (|\text{psg}(p, \tau_1)| + |\text{psg}(p, \tau_2)|) \cdot |\text{psg}(p, \sigma)|. \end{aligned} \quad (1)$$

Lemma 1. *Let $(D; T)$ be a proper drawing of a double edge, and consider a proper colouring of its faces. Let f_α and f_ω be two (not necessarily distinct) faces of $(D; T)$, let $\lambda_\alpha \in f_\alpha$ and $\lambda_\omega \in f_\omega$, $\lambda_\alpha \neq \lambda_\omega$, be two points in the plane, and let $\sigma : [0, 1] \rightarrow \mathbb{R}^2$ be a curve with $\sigma(0) = \lambda_\alpha$ and $\sigma(1) = \lambda_\omega$, such that $T' = \{\tau_1, \tau_2, \sigma\}$ constitutes a proper drawing.*

Then we have:

$$\begin{aligned} \chi(\tau_1, \sigma; T') &\equiv \chi(\tau_2, \sigma; T') \pmod{2} \text{ if } f_\alpha \text{ and } f_\omega \text{ are of the same color,} \\ \chi(\tau_1, \sigma; T') &\not\equiv \chi(\tau_2, \sigma; T') \pmod{2} \text{ if } f_\alpha \text{ and } f_\omega \text{ are of different colors.} \end{aligned}$$

Proof. Write the set of parameters t for which σ intersects either τ_1 or τ_2 (or both) in ascending order, i.e.,

$$\{t_1 < t_2 < \cdots < t_{k-1}\}.$$

Set $t_0 = 0$ and $t_k = 1$, $f_0 = f_\alpha$ and $f_{k+1} = f_\omega$, and observe that this determines a *decomposition* of the curve σ into points $\sigma(t_i)$ and *curve segments* $\sigma_i : (t_{i-1}, t_i) \rightarrow f_i$, where f_i is a *face* of $(D; T)$:

$$\sigma \simeq (\lambda_0 = \sigma(t_0)) \xrightarrow{\sigma_1} \sigma(t_1) \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_k} (\sigma(t_k) = \lambda_1). \quad (2)$$

For each $i \in [k-1]$ let $d_i := |\text{psg}(\sigma(t_i), \tau_1)| + |\text{psg}(\sigma(t_i), \tau_2)|$. It is easy to see that

- f_i and f_{i+1} are of the *same* colour if and only if d_i is *even*,
- f_i and f_{i+1} are of *different* colours if and only if d_i is *odd*.

Observe that according to (1),

$$\chi(\tau_1, \sigma; T') + \chi(\tau_2, \sigma; T') = \sum_{i=1}^{k-1} d_i,$$

and the parity of this sum depends only on the number of points with d_i *odd*. By the above observation, this is also the number of “colour changes” in the sequence of segments (2), whence the assertion follows. \square

Proof of Proposition 2. For $i = 1, 2, \dots, m$, we shall successively replace $\tau_i'' \in T''$ by $\tau' \in T'$ and apply an appropriate sign-modification to the weight function.

Assume we have already replaced the first $i-1$ curves and now want to replace τ_i'' . We may assume that the set of curves

$$T'' \sqcup \{\tau_i'\}$$

is properly intersecting (if this is not the case, we can apply “small local modifications” to τ'_i , such that both sets of curves $T'' \sqcup \{\tau'_i\}$ and T' are properly crossing). Clearly, $T = \{\tau'_i, \tau''_i\}$ gives a proper drawing $(D; T)$ of a double edge. Partition all points in $(G; T'')$ *except* the endpoints of τ''_i according to the color of the face they belong to in $(D; T)$, and consider the *smaller* one of these two partition classes: As long as this class contains some vertex v , drag the edge τ''_i over v (in the proper drawing constituted by $T'' \sqcup \{\tau'_i\}$) and apply the according sign-modification

$$\omega(e_i) \mapsto -\omega(e_i).$$

Observe that this operation “implicitly moves v into the other partition class” (but does not change the partition class for any other vertex). Clearly, after finitely many steps we have reached a state where *all* endpoints in $(G; T'')$ *except* the endpoints of τ''_i belong to faces of the *same* colour (in the proper drawing of the double edge), so by Lemma 1 we may replace τ''_i by τ'_i without changing the number of crossings of τ''_i with *any* other edge by this replacement, whence the assertion follows. \square

Obviously, Theorem 2 follows immediately from Propositions 1 and 2.

5 An illustrating example

Our attempt to give an exact description made our exposition lengthy, thus maybe obscuring the beauty and simplicity of the main argument: Thinking of a proper drawing as a web of “infinitely thin, ductile and flexible strings” which can be “transformed”, the main observation is that each *transition of some vertex v through some edge e* in the course of such transformation has to be accounted for by *changing the sign of the weight* of edge e , in order to leave the signed generating function of perfect matchings unchanged.

On the other hand, there are simple situation of double crossings or self-crossings of edges (see the upper pictures in Figure 7) which we may “straighten out” (see the lower pictures in Figure 7) without changing the sign of *any* perfect matching. Together, this gives a simple algorithm for transforming one proper drawing into another, while properly accounting for the sign-changes so that the signed generating function of perfect matchings remains

unchanged. Instead of trying a formal description of this algorithm, we illustrate it by an example, see Figure 8: We start with the proper drawing of $K_{3,3}$ shown in the left picture of Figure 3, and illustrate the steps for transforming this to the proper drawing shown in the right picture of Figure 3.

The left upper picture shows a “deformation” of T where vertices v_2 , v_4 and v_6 already “have arrived” at their desired positions (according to the hexagonal configuration in T'). Now we want to drag v_5 to its desired position along the path indicated by the dashed arrow. This involves a transition of v_5 through edge $e_{3,6}$, whose weight should therefore change its sign.

The upper picture in the middle shows the result of this operation. Now we want to drag v_1 to its desired position along the path indicated by the dashed arrow. This involves a transition of v_1 through edge $e_{2,3}$, whose weight should therefore change its sign. Note that after performing this operation, we may “straighten out” the crossings of edges $e_{4,5}$ with $e_{3,6}$ and $e_{1,4}$ with $e_{2,3}$ (according to the left pictures in Figure 7), and the crossings of edges $e_{5,6}$ and $e_{3,6}$ (according to the right picture in Figure 7).

The right upper picture shows the result of these operations: Observe that we may now “straighten out” crossings of edges $e_{2,3}$ with $e_{1,6}$ and $e_{1,2}$ with $e_{2,3}$. Now we want to drag v_3 to its desired position along the path indicated by the dashed arrow.

The left lower picture shows the result of these operations: By dragging edge $e_{1,6}$ over vertex v_4 and straightening out the crossing with $e_{1,4}$ thus introduced, we arrive at the lower picture in the middle. This involves a transition of v_4 through edge $e_{1,6}$, whose weight should therefore change its sign. Dragging edge $e_{3,6}$ over vertices v_4 and v_5 and straightening out the crossings with edges $e_{5,6}$, $e_{4,5}$ and $e_{3,4}$ gives the right lower picture: Clearly, by dragging edge $e_{2,5}$ over vertices v_3 and v_4 we arrive at the right proper drawing in Figure 3.

References

- [1] P.W. Kasteleyn. Graph theory and crystal physics. In F. Harary, editor, *Graph Theory and Theoretical Physics*, chapter 2, pages 43–110. Academic Press, 1967.

Figure 7: Straightening out of edges.

Consider the situation depicted in the three upper pictures, where the boundary of the square-shaped face does not contain any vertices or crossings except the ones actually shown in the pictures: Then the edges might involved can be “straightened out” to obtain the three lower pictures, respectively, without changing the signs of any perfect matching.

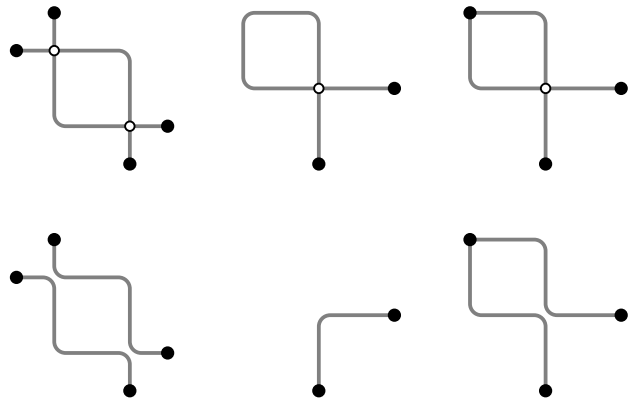
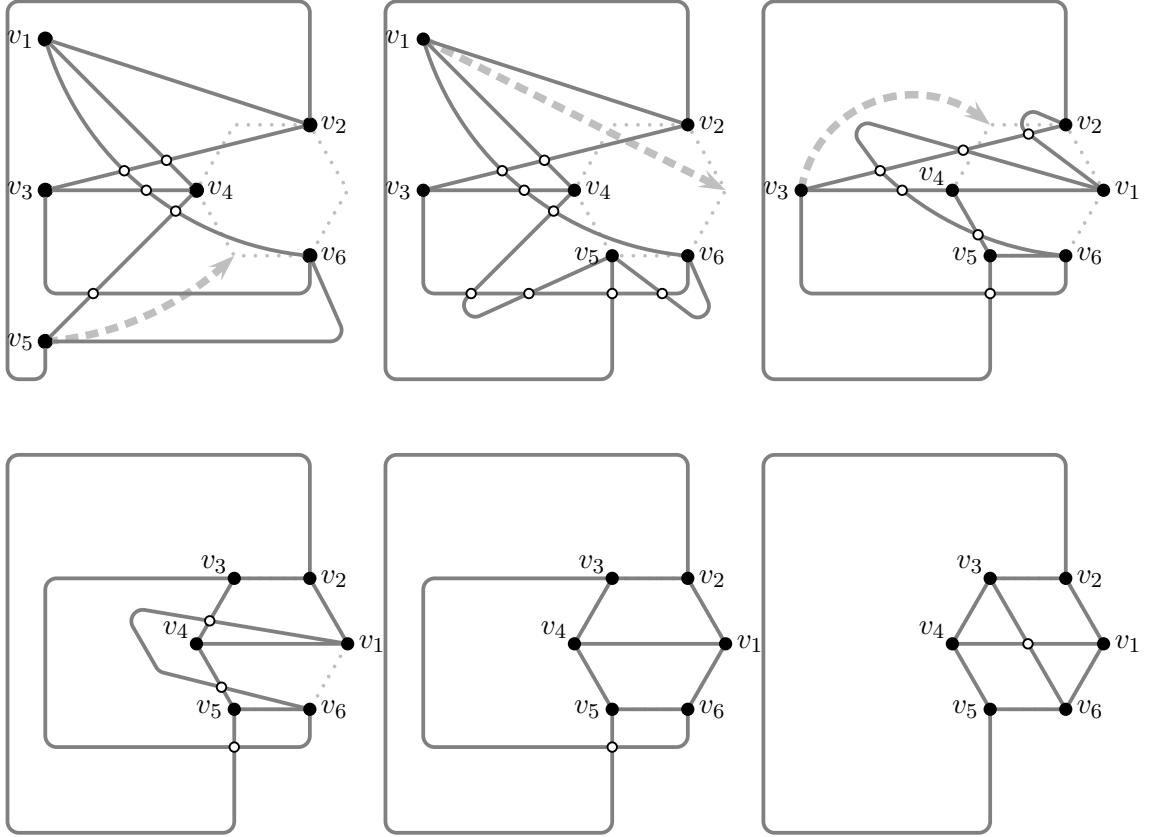


Figure 8: Example: Transformation of proper drawings.
 Consider the proper drawings of $K_{3,3}$ depicted in Figure 3: We want to transform the left proper drawing T to the right proper drawing T' .



- [2] David E. Speyer. Variations on a theme of Kasteleyn, with application to the totally nonnegative Grassmannian. *Electr. J. Comb.*, 23:P2.24, 2016.
- [3] John R. Stembridge. Nonintersecting paths, pfaffians and plane partitions. *Adv. Math.*, 83:96–131, 1990.